Unit-5

Numerical Integration:

We know that $\int_a^b f(x)\,dx$ represents the area between $y = f(x)$, x-axis and the ordinates $x=a$ and $x=b$. This integration is possible only if the $f(x)$ is explicitly given and if it is integrable. The problem of numerical integration can be stated as follows:

Given a set of $(n+1)$ paired values (x_i, y_i), $i=0,1,2,\ldots n$ of the function $y=f(x)$, where $f(x)$ is not known explicitly, it is required to compute

$$\int_{x_0}^{x_n} y \cdot dx.$$

As we did in the case of interpolation or numerical differentiation, we replace $f(x)$ by an interpolating polynomial $P_n(x)$ and obtain $\int_{x_0}^{x_n} P_n(x)\,dx$ which is approximately taken as the value for

$$\int_{x_0}^{x_n} f(x)\,dx.$$

* Trapezoidal rule :

By putting $n=1$, in the quadrature formula (ie, there are only two paired values and interpolating polynomial is linear).

$$\int_{x_0}^{x_0+h} f(x)\,dx = h\left[1.\,y_0 + \frac{1}{2}\Delta y_0\right]$$ since other differences do not exists if $n=1$.

$$= h\left[y_0 + \frac{1}{2}(y_1 - y_0)\right]$$

$$= \frac{h}{2}(y_0 + y_1)$$

$$\int_{x_0}^{x_n} f(x)\,dx = \int_{x_0}^{x_0+nh} f(x).\,dx$$

$$= \int_{x_0}^{x_0+h} f(x)\,dx + \int_{x_0+h}^{x_0+2h} f(x)\,dx + \cdots +$$

$$\int_{x_0+(n-1)h}^{x_0+nh} f(x)\,dx$$

$$= \frac{h}{2}(y_0+y_1) + \frac{h}{2}(y_1+y_2) + \cdots + \frac{h}{2}(y_{n-1}+y_n)$$

$$= \frac{h}{2}\left[(y_0+y_n) + 2(y_1+y_2+y_3+\cdots+y_{n-1})\right]$$

$$= \frac{h}{2}\left[(\text{Sum of the first and last ordinates}) + 2(\text{sum of the remaining ordinates})\right]$$

This is known as <u>Trapezoidal rule</u>.

* Simpson's one-third rule

Setting $n=2$ in Newton-cote's quadrature formula, we have

$$\int_{x_0}^{x_n} f(x)\,dx = h\left[ny_0 + \frac{n^2}{2}\Delta y_0 + \frac{1}{2}\left(\frac{n^3}{3}-\frac{n^2}{2}\right)\Delta^2 y_0 + \frac{1}{6}\left(\frac{n^4}{4}-n^3+n^2\right)\Delta^3 y_0 + \cdots\right] \rightarrow (1)$$

Setting $n=2$,

$$= h\left[2y_0 + \frac{4}{2}\Delta y_0 + \frac{1}{2}\left(\frac{8}{3}-\frac{4}{2}\right)\Delta^2 y_0\right]$$

(since other terms vanish).

$$= h\left[2y_0 + 2(y_1-y_0) + \frac{1}{3}(E-1)^2 y_0\right]$$

$$= h\left[2y_0 + 2y_1 - 2y_0 + \frac{1}{3}(y_2 - 2y_1 + y_0)\right]$$

$$= h\left[\frac{1}{3}y_2 + \frac{4}{3}y_1 + \frac{1}{3}y_0\right]$$

$$= \frac{h}{3}(y_2 + 4y_1 + y_0)$$

lllrly,

$$\int_{x_2}^{x_4} f(x)\,dx = \frac{h}{3}(y_2 + 4y_3 + y_4)$$

$$\int_{x_i}^{x_{i+2}} f(x)\,dx = \frac{h}{3}(y_i + 4y_{i+1} + y_{i+2}).$$

If n is an even integer, last integral will be

$$\int_{x_{n-2}}^{x_n} f(x)\,dx = \frac{h}{3}(y_{n-2} + 4y_{n-1} + y_n)$$

Adding all these integrals, if n is an even positive integer, that is, the number of ordinates $y_0, y_1, \dots, y_n$ is odd, we have

$$\int_{x_0}^{x_n} f(x)\,dx = \int_{x_0}^{x_2} f(x)\,dx + \int_{x_2}^{x_4} f(x)\,dx + \dots + \int_{x_{n-2}}^{x_n} f(x)\,dx$$

$$= \frac{h}{3}\Big[(y_0 + 4y_1 + y_2) + (y_2 + 4y_3 + y_4) + \dots + (y_{n-2} + 4y_{n-1} + y_n)\Big]$$

$$= \frac{h}{3}\left[(y_0 + y_n) + 2(y_2 + y_4 + \cdots) + 4(y_1 + y_3 + \cdots)\right]$$

$= \frac{h}{3}$ [Sum of the first and last ordinates

+ 2 (sum of remaining odd ordinates)

+ 4 (sum of even ordinates)].

Note : Though y_2 has suffix even, it is the third ordinate (odd).

* Simpson's three-eights rule

Putting $n = 3$ in Newton-cotes formula (eqn. ①)

we have

$$\int_{x_0}^{x_3} f(x)\,dx = h\left[3y_0 + \frac{9}{2}\Delta y_0 + \frac{1}{2}\left(\frac{9}{2}\right)\Delta^2 y_0 + \frac{1}{6}\left(\frac{81}{4} - 27 + 9\right)\Delta^3 y_0\right].$$

$$= h\left[3y_0 + \frac{9}{2}(y_1 - y_0) + \frac{9}{4}(E-1)^2 y_0 + \frac{3}{8}(E-1)^3 y_0\right]$$

$$= h\left[3y_0 + \frac{9}{2}y_1 - \frac{9}{2}y_0 + \frac{9}{4}(y_2 - 2y_1 + y_0) + \frac{3}{8}(y_3 - 3y_2 + 3y_1 - y_0)\right]$$

$$= \frac{3h}{8}\left[y_3 + 3y_2 + 3y_1 + y_0\right]$$

If n is a multiple of 3,

$$\int_{x_0}^{x_0+nh} f(x)\,dx = \int_{x_0}^{x_0+3h} f(x)\,dx + \int_{x_0+3h}^{x_0+6h} f(x)\,dx + \cdots + \int_{x_0+(n-3)h}^{x_0+nh} f(x)\,dx$$

$$= \frac{3h}{8}\Big[(y_0+3y_1+3y_2+y_3) + (y_3+3y_4+3y_5+y_6) + \cdots + (y_{n-3}+3y_{n-2}+3y_{n-1}+y_n)\Big]$$

$$= \frac{3h}{8}\Big[(y_0+y_n) + 3(y_1+y_2+y_4+y_5+\cdots+y_{n-1}) + 2(y_3+y_6+y_9+\cdots+y_n)\Big] \rightarrow (2)$$

Equation (2) is called Simpson's three-eights rule which is applicable only when n is a multiple of 3.